THEORY OF THE SUPERSONIC PART OF A NONDISSIPATIVE BOX-TYPE ACCELERATOR

A. I. Morozov

Zhurnal Prikladnoi Mekhaniki i Tekhnicheskoi Fiziki, No. 2, pp. 30–35, 1966

This paper calculates the two-dimensional flow of a two-component cold nondissipative plasma. The condition is found for which a nondissipative plasma can escape from a magnetic field.

A distinction must be made between those plasma accelerators which operate at high densities $n \gtrsim 10^{16}\text{cm}^{-3}$ and those which operate at low densities ($n \lesssim 10^{14}$ cm^{-3}).

In the first case the acceleration of ions may occur not only as a result of internal electric fields, but also as a result of the collision of ions with each other (ordinary thermal acceleration) or the collision of ions with electrons moving as a result of the Hall* effect along the accelerator channel.

In the second case the only mechanism capable of accelerating the ions is the internal longitudinal electric field [1, 2]. This treatment excludes those systems [3], in which the ions are accelerated as a result of kinetic instabilities.

The present paper treats the acceleration of a low-density plasma. A box-type accelerator has been chosen by way of example, since in this case the peculiarities of low-density accelerating systems are particularly marked.

Three conditions must be fulfilled in order to create a box-type accelerator operating at low densities.

Firstly, we must make use of segmented electrodes, since otherwise it is impossible to create in the space inside the accelerator electric fields of the type (2.11) necessary for accelerating the plasma. For continuous electrodes the structure of the electric field in the space inside the accelerator results from the superposition and interaction of exceedingly complex processes at the boundaries and in the vicinity of the electrodes.

Secondly, if the system under consideration is intended to produce high velocities, then it is necessary to create conditions under which there is no interaction of the fast particle flux with the walls. In other words, the current in the system must be controlled electromagnetically and not by means of the walls, which is the case in ordinary gasdynamic nozzles.

It should be noted that in high-velocity plasma accelerators which produce ions with energies $\gtrsim 100$, it is essential to eliminate the interaction of the current with the walls at higher densities also.

Thirdly and lastly, conditions must be created which ensure the escape of the plasma from the magnetic field.

It was shown in [4] that the magnetic field is in fact "frozen" into the electronic component of the plasma, and so a necessary condition for a compensating current to escape from the magnetic field is that the exchange parameter**

$$\xi \gg 1$$

should be large.

The present paper develops the theory of the supersonic part of a box-type accelerator. It should be noted that many papers [5–8] have been devoted to the theory of the box-type accelerator. These, however, were written on the assumption either that the medium is incompressible or for the quasi-one-dimensional approximation in which the flow is of necessity given a particular geometry. These assumptions are, of course, very far removed from reality, at least as far as the production of high-velocity flows is concerned.

1. Initial equations. If precautions are taken so that the plasma flux is kept away from the walls and if its density is not very large, then dissipative processes may be neglected. If, in addition to this, the flow of ions may be taken to be substantially supersonic, then the steady-state ion motion will be described by the equations

$$\operatorname{div} n\mathbf{v}_i = 0, \qquad m\frac{d\mathbf{v}_i}{dt} = e\left(\mathbf{E} + \frac{1}{c}\mathbf{v}_i \times \mathbf{H}\right). \tag{1.1}$$

Here $\mathbf{v}_i$ is the ion velocity, **E**, **H** are the electric and magnetic field strengths, e, m are the charge and mass of an ion, n is the ion concentration, which is taken to be equal to the electron concentration. We assume almost complete ionization and neglect the presence of neutrals.

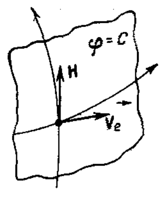

Fig. 1

Similarly, if the electron temperature, measured in eV is small in comparison with the potential difference applied to the accelerator, then apart from the layers close to the electrodes we may take the following system of equations as a good approximation for the electron component

$$\operatorname{div} n\mathbf{v}_e = 0, \qquad \mathbf{E} + \frac{1}{c}\mathbf{v}_e \times \mathbf{H} = 0. \tag{1.2}$$

The Hall effect is allowed for in the second equation (1.2), where $\mathbf{v}_e$ appears instead of $\mathbf{v}_i$. It follows from the second equation (1.2), firstly, that the lines of force γ of the magnetic field are equipotentials of the electric field

$$\varphi = \varphi(\gamma) \tag{1.3}$$

and, secondly, that the electron drift takes place over the equipotential surfaces (Fig. 1). The problem is to calculate the motion of ions and electrons described by Eqs. (1.1) and (1.2) under the conditions of a box-type accelerator.

In the box-type accelerator (Fig. 2) the magnetic field may be taken to be given, i.e., we may neglect

---

*In is natural to call this method of acceleration ohmic, since it is due to the ohmic resistance of the plasma.

**In accordance with [4] the exchange parameter is understood to be the ratio of the discharge current I_0 to the ion current I_i escaping from the accelerator $\xi = I_0 / I_i$, $I_i = eq/m$, where q is the mass flow rate of working medium.

the self magnetic field of currents flowing in the plasma. If we confine ourselves to the case of flat magnetic fields, when all the lines of force lie in the planes y = const, then they may be described by one component of the vector potential

$$A_y(x, z) = \int^x H_0(x)\,dx - H_0'(x)\frac{z^2}{2}\,. \qquad (1.4)$$

Here $H_0(x)$ is the field strength in the plane of symmetry z = 0.

In the particular case when the magnetic field H_0 decreases linearly,

$$H_0 = H_{00}(1 - x/L), \qquad (1.5)$$

the component A_y is equal to

$$A_y = -{}^1/_2 H_{00} L\psi, \qquad \psi \equiv (1 - x/L)^2 - z^2/L^2. \; (1.6)$$

The equation of the lines of force has the form [9]

$$\psi = \text{const}, \qquad y = \text{const}.$$

Consequently, in this case Eq. (1.3) may be written in the form

$$\varphi = \varphi(\psi, y). \qquad (1.7)$$

For a given magnetic field the function $\varphi(x, y, 0)$ may be arbitrary. However, it will be clear in what follows that it is convenient to specify the function $\varphi(x, 0, 0)$, since for z = 0 we must choose φ to be a function of y such that the plasma flow is of the type required.

In order to solve the fairly complex system (1.1), (1.2), (1.7), we have recourse to the approximation of ion-optics, i.e., we specify one "basic" trajectory with a known law of variation of ion velocity along it and seek all quantities in the form of series of powers of the departure from the basic trajectory. This is valid since the transverse dimensions of the box-type accelerator are small compared with its length. We take the x axis as the basic trajectory. Then the quantities may be expanded as follows:

$$(\mathbf{v} \equiv \mathbf{v}_i, \; \mathbf{u} \equiv \mathbf{v}_e)$$

$$\begin{aligned}
v_x &= \mathbf{v}_1(x) + yv_{12}(x) + zv_{13}(x) + \ldots \\
u_x &= u_1(x) + yu_{12}(x) + zu_{13}(x) + \ldots \\
v_y &= yv_{22}(x) + zv_{23}(x) + \ldots \\
u_y &= u_2(x) + yu_{22}(x) + zu_{23}(x) + \ldots \\
v_z &= yv_{32}(x) + zv_{33}(x) + \ldots \\
u_z &= yu_{32}(x) + zu_{33}(x) + \ldots \\
n &= n_1(x) + yn_2(x) + zn_3(x) + \ldots \qquad (1.8)
\end{aligned}$$

Similarly, we expand relation (1.7) in powers of y:

$$\varphi = \varphi_0(\psi) + y\varphi_1(\psi) + {}^1/_2 y^2\varphi_2(\psi) + \ldots \; . \quad (1.9)$$

Setting expansion (1.8), (1.9) in Eqs. (1.1), (1.2), we obtain a system of equations for the coefficients of the expansion.

It turns out that in addition to very complicated solutions, this system has the following simple solution with an accuracy to terms of the first order of smallness inclusive:

$$\begin{aligned}
v_x &= v_1 + y\frac{e}{mc}H_0(x) + 0 + \ldots, \\
u_x &= v_1 + y\frac{e}{mc}H_0(x) + 0 + \ldots, \\
v_y &= 0 + 0 + 0 + \ldots, \\
u_y &= \frac{c}{H_0}\frac{\partial\varphi_0}{\partial x}\Big|_{z=0} - y\frac{(H_0v_1)'}{H_0} + 0 + \ldots, \\
v_z &= 0 + 0 + zv_{33} + \ldots, \\
u_z &= 0 + 0 + zv_{33} + \ldots, \qquad (1.10)
\end{aligned}$$

$$\frac{m}{e}\frac{v_1^2}{2} + \varphi_0\big|_{z=0} \equiv \varphi_{00} = \text{const}, \qquad \varphi_1\big|_{z=0} = -\frac{v_1H_0(x)}{c},$$

$$n_3 = 0, \qquad \frac{\partial}{\partial x}n_1v_1 + n_1v_{33} = 0, \qquad n_2u_2 + n_1u_{22} = 0\,.$$

The quantity v_{33} is determined by a Riccati-type equation

$$\frac{m}{e}(v_1v_{33}' + v_{33}^2) = H_0'\frac{d\varphi_0}{d\psi}\Big|_{z=0}. \qquad (1.11)$$

The particular solution which has been found is of interest because it describes a plasma flow such that the ion velocity has only the longitudinal components v_x, v_z and the current density $\mathbf{j} = en(\mathbf{v} - \mathbf{u})$ has only one component j_y.

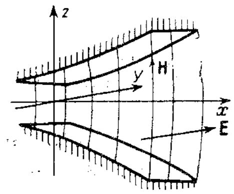

Fig. 2

2. Investigation of the solution (1.10). Our attention is first of all drawn to the fact [see (1.10)] that the plasma flux is broadened along the z axis. This effect is due to the convexity of the magnetic lines of force. In one-dimensional theories, however, it was not allowed for and was formally "suppressed," for example by walls z = const confining the flow with respect to z. Such a form of "suppression" can be assumed only for the acceleration of a dense plasma to velocities which are not very high and is inapplicable to the acceleration of a rarefied plasma to high velocities. It is a simple matter to calculate the spread of the flux in the case when the magnetic field varies linearly according to (1.5), and the potential $\varphi_0(\psi)$ is a linear function of ψ:

$$\varphi_0 = \varphi_{00}\psi, \qquad \varphi_0\big|_{z=0} = \varphi_{00}\left(1 - \frac{x}{L}\right)^2. \qquad (2.1)$$

Then on the basis of (1.10) the variation of velocity v_1 along the x axis is given by the formula

$$v_1 = v_m\sqrt{1 - (1 - x/L)^2}\,, \qquad \max v \equiv v_m \equiv \left(\frac{2e\varphi_{00}}{m}\right)^{1/2}. \; (2.2)$$

Setting (2.1) and (2.2) in (1.11), we find the expression for v_z

$$v_z = z\frac{v_m}{L}\frac{\exp(2\arccos\zeta)-1}{\exp(2\arccos\zeta)+1}, \qquad \zeta \equiv 1-\frac{x}{L}, \qquad (2.3)$$

and at the same time with the help of the equation

$$dz / dx = v_z / v_1, \qquad (2.4)$$

we determine the boundary of the beam in the xz plane:

$$z_s = z_0 \operatorname{ch}(\arccos\zeta). \qquad (2.5)$$

Here z_0 is the beam width at the entrance to the accelerator channel. The beam width at the exit, i.e., for x = L (ζ = 0) is equal to

$$(z_s)_{\max} = z_0 \operatorname{ch} {}^1/_2\pi \approx 2.63 z_0. \qquad (2.6)$$

Knowing how the beam broadens, we may find the variation of density in the y = 0 plane. It follows from the solution (1.10) for v_z that

$$n_1 v_1 z_s = \text{const}. \qquad (2.7)$$

Thus

$$n_1 = \frac{\text{const}}{z_0[\operatorname{ch}(\arccos\zeta)]\, v_m \sqrt{1-\zeta^2}}. \qquad (2.8)$$

The last equation of (1.10) shows that the plasma density depends on y. In particular, if the field decreased linearly

$$n = n_1\left[1 + y\frac{eH_{00}}{v_m mc}\frac{(1-2\zeta^2)}{\zeta(1-\zeta^2)^{1/2}}\right]. \qquad (2.9)$$

It was noted above that the solution obtained describes a plasma flow which has only one y component of current. This current is due to the drift of electrons in the y direction under the action of the longitudinal electric field E_x. The fact that the drift velocity u_y is a function of y (1.10) leads to a variation of the plasma density in the direction of the y axis [see (2.9)].

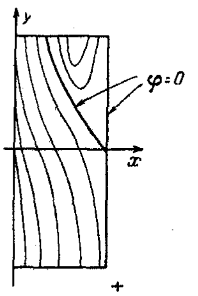

Fig. 3

For the case (1.5), (2.1) it is not difficult to see that the magnitude of the current density in the cross section y = 0 is equal to

$$j = e\frac{v_m^2 mc}{LeH_{00}} n_1 = e n_1 v_m \frac{R_{i0}}{L}, \qquad R_{i0} \equiv \frac{v_m mc}{eH_{00}}. \qquad (2.10)$$

If we take into account that $n_1 \to \infty$ for $x \to 0$, then $j \to \infty$ at the entrance.

The electric potential in the channel is described by the expression

$$\varphi = \varphi_0(x, 0) - \frac{1}{c} y v_1 H_0(x) + \ldots. \qquad (2.11)$$

The first term of this expression describes the longitudinal electric field which accelerates the ions. The second term describes the transverse field which balances the Lorentz force and ensures that $v_y = 0$, as well that the longitudinal velocities of ions and electrons are equal, i.e., the absence of a longitudinal current. If we let b denote the channel width, then the potential V(x) between the electrodes will be equal to

$$V(x) = b\frac{v_1(x) H_0(x)}{c}.$$

For the case (1.5), (2.1)

$$V(x) = V_m 2\zeta\sqrt{1-\zeta^2}, \text{ where } V_m \equiv {}^1/_4 b v_m H_{00}$$

the maximum value of the potential for

$$x_m = L(\sqrt{2}-1)/\sqrt{2} \approx 0.3L.$$

The pattern of potential distribution in the xy plane for cases close to (1.5), (2.1) has the form depicted in Fig. 3. The nonmonotonic dependence of φ on x for y > 0 is explained by the rapid decrease in φ for $x \to L$. It is practically feasible to make systems* which operate for y < 0.

If the exchange parameter [4] is calculated for a given system it is equal to

$$\xi = \frac{I_0}{I_i} = \frac{1}{eN}\int_0^L j_y f dx = \frac{1}{b}\int_0^{'} \frac{dv}{eH_0/mc} \qquad (2.12)$$

in the general case (see 1.10).

Here f(x) is the height of the channel. In the particular case of (1.5) and (2.1)

$$\xi = \frac{1}{b}R_{i0}\frac{\pi}{2}, \qquad R_{i0} \equiv \frac{v_m mc}{eH_{00}}. \qquad (2.13)$$

In order to ensure the escape of the plasma from the magnetic field when there is no dissipation it is essential to make $\xi \gg 1$ [4]. To determine the minimal value of ξ for this or that concrete system by theoretical means is not easy and is outside the limits of the present paper.

If some a priori value ξ_0 is taken, for example ξ_0 = 10, then for a given discharge rate N we automatically have an expression for the discharge current

$$I_0 = \xi_0 eN.$$

*For the coordinate system and functions (1.5) and (2.1) selected here.

If the velocity of flow and channel width are given at the same time, then the field strength is found with the help of (2.13):

$$H_{00}=\frac{v_m\pi}{2eb}\frac{mc}{\xi_0}.$$

We shall now consider the question of the small parameters in terms of which the expansion (1.8) is carried out.

The solution (1.10) shows that everywhere, with the exception of the neighborhood of the singular point H = 0, the expansion may be made in terms of the quantities

$$\mu_1=y\frac{cH_0}{mcv_1},\qquad \mu_2=\frac{Z}{L}.$$

Hence it is clear that expansion (1.8) will be valid over the main part of the channel, i.e., with the exception of the entrance $v\to 0$ and the exit $H\to 0$, only if $\mu_1 \ll 1$ and $\mu_2 \ll 1$. However, formulas (2.12), (2.13) show that $\mu_1 \sim 1/\xi$ and that consequently we may expect a smoothly varying solution only for $\xi \gg 1$. In the opposite case, i.e., for $\xi \leqslant 1$, a box-type accelerator will not, in all probability, be able to operate without dissipation, or else the flow will be much deformed. In this connection it is natural to take $\xi \gg 1$ as the condition for nondissipative escape of the plasma from the magnetic field. The parameter has a natural connection with the geometry of the magnetic field, and the requirement that it be small is quite obvious.

In conclusion we note that the supposition that the self-induced magnetic field is small, made at the start of the calculations, is valid if

$$\frac{H_0^2}{4\pi mn}\gg v_m^2.$$

Further, it is important to keep in mind that expression (2.11) describing the potential distribution does not satisfy Laplace's equation in the general case [10]. Thus the accelerator for which the calculations have been made is basically a plasma accelerator. It is essential for its normal functioning that the Debye radius should be considerably less than the dimensions of the system. This condition is, however, fulfilled at small densities $\leqslant 10^{10}$ cm^{-3}.

Finally, we note that neglecting the thermal pressure and ohmic resistance of the plasma amounts to neglecting the terms $\sim 1/R_m$, $1/M^2$, where R_m is the magnetic Reynolds number, and M is the Mach number.

In conclusion the author is grateful to A. I. Bugrov, L. E. Kalikhman, and L. S. Solov'ev for discussing the questions raised.

REFERENCES

1. A. I. Morozov, "The acceleration of a plasma by a magnetic field," ZhETF, vol. 32, p. 305, 1957.

2. A. I. Morozov and L. S. Solov'ev, "Plane flow of an ideally conducting compressible fluid taking the Hall effect into account," ZhETF, vol. 34, p. 1141, 1964.

3. M. V. Nezlin and A. M. Solntsev, "The acceleration of ions in plasma beams," ZhETF, vol. 45, p. 840, 1963.

4. A. I. Morozov and L. S. Solov'ev, "A certain similarity parameter in the theory of plasma flow," DAN SSSR, vol. 150, p. 725, 1965.

5. G. Wood and A. Carter, Considerations of NASA relating to the construction of a continuous plasma accelerator," collection: Ion, Plasma, and Arc Rocket Motors [Russian translation], Gosatomizdat, 1961.

6. A. B. Vatazhin, "Some two-dimensional problems of current distribution in an electrically conducting medium moving along a channel in a magnetic field," PMTF, no. 2, p. 39, 1963.

7. G. Sutton and A. Carlson, "End effects in inviscid flow in a magnetohydrodynamic channel," collection: The Magnetohydrodynamic Method of Energy Conversion [Russian translation], Fizmatgiz, 1963.

8. L. E. Kalikhman, The Elements of Magnetogasdynamics [in Russian], Atomizdat, 1964.

9. A. I. Morozov and L. S. Solov'ev, "Geometry of the magnetic field," collection: Problems of Plasma Theory [in Russian], Gosatomizdat, no. 2, 1963.

10. A. I. Morozov, "The focusing of cold quasineutral beams in electromagnetic fields," DAN SSSR, vol. 163, p. 1363, 1965.

7 June 1965 Moscow